Physics 566: Quantum Optics I
Problem Set 4, Solutions

Problem 1: Some Algebra with Density Matrices

(ai) $\mathrm{Tr}(\hat{A}) = \sum_i \langle e_i|\hat{A}|e_i\rangle = \sum_{i,j,k} \langle e_i|\underbrace{f_j\rangle\langle f_j}_{\text{complete set of another basis}}|\hat{A}|f_k\rangle\overbrace{\langle f_k|e_i\rangle}^{\delta_{jk}} = \sum_j \langle f_j|\hat{A}|f_j\rangle$ Trace is the sum of diagonal matrix element in any basis.

(aii) $\mathrm{Tr}(\hat{A}|\psi\rangle\langle\phi|) = \sum_i \langle e_i|\hat{A}|\psi\rangle\langle\phi|e_i\rangle = \sum_i \langle\phi|e_i\rangle\langle e_i|\hat{A}|\psi\rangle = \langle\phi|\hat{A}|\psi\rangle$

(aiii) $\mathrm{Tr}(\hat{A}\hat{B}\hat{C}) = \sum_{ij}\langle e_i|\hat{A}\hat{B}|e_j\rangle\langle e_j|\hat{C}|e_i\rangle = \sum_{ji}\langle e_j|\hat{C}|e_i\rangle\langle e_i|\hat{A}\hat{B}|e_j\rangle = \sum_j \langle e_j|\hat{C}\hat{A}\hat{B}|e_j\rangle =$

$= \mathrm{Tr}(\hat{C}\hat{A}\hat{B})$ (also $= \mathrm{Tr}(\hat{B}\hat{C}\hat{A})$: cyclic permutations!

(b) For a qubit $\hat{\rho} = \frac{1}{2}(\hat{\mathbb{1}} + \vec{Q}\cdot\hat{\vec{\sigma}})$ where $\vec{Q}$ is the Bloch vector, $\vec{Q} = \langle\hat{\vec{\sigma}}\rangle$.

$$\hat{\rho}^2 = \frac{1}{4}\left(\hat{\mathbb{1}} + 2\vec{Q}\cdot\hat{\vec{\sigma}} + \sum_{ij} Q_i Q_j \hat{\sigma}_i\hat{\sigma}_j\right) \qquad \hat{\sigma}_i\hat{\sigma}_j = i\epsilon_{ijk}\hat{\sigma}_k + \delta_{ij}\hat{\mathbb{1}}$$

$$\Rightarrow \hat{\rho}^2 = \frac{1}{4}\left[(1+|\vec{Q}|^2)\hat{\mathbb{1}} + 2\vec{Q}\cdot\hat{\vec{\sigma}}\right] + \frac{i}{4}\cancel{(\vec{Q}\times\vec{Q})}^{\,0}\cdot\hat{\vec{\sigma}}$$

$$\Rightarrow \mathrm{Tr}(\hat{\rho}^2) = \frac{1}{4}\left[(1+|\vec{Q}|^2)\underbrace{\mathrm{Tr}(\hat{\mathbb{1}})}_{=2} + 2\vec{Q}\cdot\underbrace{\mathrm{Tr}(\hat{\vec{\sigma}})}_{=0}\right] \Rightarrow \boxed{\mathrm{Tr}(\hat{\rho}^2) = \frac{1}{2}(1+|\vec{Q}|^2)}$$

General $\underset{\text{maximally mixed}}{\frac{1}{d}} \leq \mathrm{Tr}(\hat{\rho}^2) \leq \underset{\text{pure}}{1}$, where d is the dimension of the Hilbert space.

Here $d=2 \Rightarrow \mathrm{Tr}(\hat{\rho}^2_{pure}) = 1 = \frac{1}{2}(1+|\vec{Q}_{pure}|^2) \Rightarrow \boxed{|\vec{Q}_{pure}| = 1}$

$\mathrm{Tr}(\hat{\rho}^2_{max\,mix}) = \frac{1}{2} = \frac{1}{2}(1+|\vec{Q}_{max\,mixed}|^2) \Rightarrow \boxed{|\vec{Q}_{max\,mixed}| = 0}$

(c) Consider a Mach-Zender interferometer with "dual-rail" encoding of a qubit. The transformation of input to output is the unitary map $\hat{U}(\phi) = e^{-i\frac{\phi}{2}\hat{\sigma}_z}$ where ϕ is the phase difference between between the two paths. Given an input state $\hat{\rho}_{in}$, the output state is $\hat{\rho}_{out} = \hat{U}(\phi)\hat{\rho}_{in}\hat{U}^\dagger(\phi)$. The probability of find $|1_a, 0_b\rangle$ at the output is then

$$P_{1_a,out} = \langle 1_a, 0_b | \hat{\rho}_{out} | 1_a, 0_b \rangle = \langle 1_a, 0_b | \hat{U}(\phi)\, \hat{\rho}_{in}\, \hat{U}^{\dagger}(\phi) | 1_a, 0_b \rangle \equiv \langle \uparrow_z | e^{-i\frac{\phi}{2}\hat{\sigma}_z}\, \hat{\rho}_{in}\, e^{i\frac{\phi}{2}\hat{\sigma}_z} | \uparrow_z \rangle$$

$$e^{-i\frac{\phi}{2}\hat{\sigma}_z} = \cos\frac{\phi}{2}\,\hat{\mathbb{1}} - i\sin\frac{\phi}{2}\,\hat{\sigma}_y$$

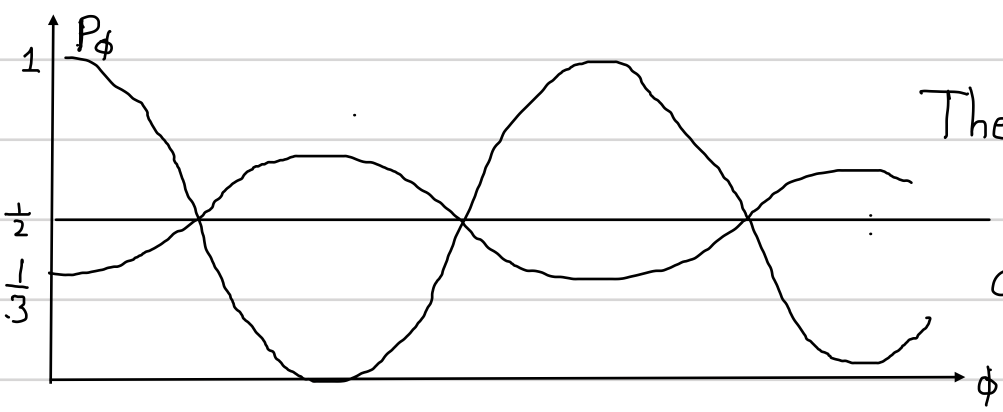
(i) $\hat{\rho}_{in} = |1_a, 0_b\rangle\langle 1_a, 0_b| \equiv |\uparrow_z\rangle\langle\uparrow_z| \Rightarrow P_{1_a 0_b} = |\langle\uparrow_z| e^{-i\frac{\phi}{2}\hat{\sigma}_z} |\uparrow_z\rangle|^2 = \cos^2\frac{\phi}{2}$

(ii) $\hat{\rho}_{in} = \frac{1}{2}|1_a, 0_b\rangle\langle 1_a, 0_b| + \frac{1}{2}|0_a, 1_b\rangle\langle 0_a, 1_b| = \frac{1}{2}\hat{\mathbb{1}} \Rightarrow P_{1_a out} = \frac{1}{2}$ (independent of ϕ)

(iii) $\hat{\rho}_{in} = \frac{1}{3}|1_a, 0_b\rangle\langle 1_a, 0_b| + \frac{2}{3}|0_a, 1_b\rangle\langle 0_a, 1_b| = \frac{1}{3}|\uparrow_z\rangle\langle\uparrow_z| + \frac{2}{3}|\downarrow_z\rangle\langle\downarrow_z|$

$$\Rightarrow P_{1_a,out} = \frac{1}{3}|\langle\uparrow_z| e^{-i\frac{\phi}{2}\hat{\sigma}_z} |\uparrow_z\rangle|^2 + \frac{2}{3}|\langle\uparrow_z| e^{-i\frac{\phi}{2}\hat{\sigma}_z} |\downarrow_z\rangle|^2$$

$$= \frac{1}{3}\cos^2\frac{\phi}{2} + \frac{2}{3}\sin^2\frac{\phi}{2} = \left[\left(\frac{1}{3}+\frac{2}{3}\right) + \left(\frac{1}{3}-\frac{2}{3}\right)\cos\phi\right]/2 = \left(1 - \frac{1}{3}\cos\phi\right)/2$$



The visibility of the fringes correlates Coherence (purity) of the state.

Problem 2: Ensemble decomposition and the density matrix for qubits

(a) Statistical mixture: $\hat{\rho} = \frac{1}{2}(1+\frac{1}{\sqrt{2}})|\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{2}(1-\frac{1}{\sqrt{2}})|\downarrow_z\rangle\langle\downarrow_z|$

Representation: Basis $\{|\uparrow_z\rangle, |\downarrow_z\rangle\}$ $\rho_{\alpha\beta} = \langle\alpha|\hat{\rho}|\alpha\rangle \Rightarrow \rho = \begin{bmatrix} \frac{1}{2}(1+\frac{1}{\sqrt{2}}) & 0 \\ 0 & \frac{1}{2}(1-\frac{1}{\sqrt{2}}) \end{bmatrix}$

Basis $\{|\uparrow_x\rangle, |\downarrow_x\rangle\}$, recall $\langle\uparrow_x|\uparrow_z\rangle = \langle\downarrow_x|\uparrow_z\rangle = \langle\uparrow_x|\downarrow_z\rangle = \frac{1}{\sqrt{2}}$, $\langle\downarrow_x|\downarrow_z\rangle = -\frac{1}{\sqrt{2}}$

$\Rightarrow \langle\uparrow_x|\hat{\rho}|\uparrow_x\rangle = \langle\downarrow_x|\hat{\rho}|\downarrow_x\rangle = \frac{1}{2}$ $\qquad \langle\uparrow_x|\hat{\rho}|\downarrow_x\rangle = \langle\downarrow_x|\hat{\rho}|\uparrow_x\rangle = \frac{1}{2\sqrt{2}}$

$$\Rightarrow \rho = \frac{1}{2}\begin{bmatrix} 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 \end{bmatrix}$$

The Bloch vector: $\vec{Q} = \mathrm{Tr}(\hat{\rho}\,\vec{\sigma}) = \frac{1}{2}(1+\frac{1}{\sqrt{2}})\,\vec{e}_z + \frac{1}{2}(1-\frac{1}{\sqrt{2}})(-\vec{e}_z) = \frac{1}{\sqrt{2}}\vec{e}_z$

(b) Another statistical mixture $\hat{\rho} = \frac{1}{2}|\uparrow_{n_1}\rangle\langle\uparrow_{n_1}| + \frac{1}{2}|\uparrow_{n_2}\rangle\langle\uparrow_{n_2}|$, $\vec{e}_1 = \frac{\vec{e}_z+\vec{e}_x}{\sqrt{2}}$, $\vec{e}_2 = \frac{\vec{e}_z-\vec{e}_x}{\sqrt{2}}$

This is not a completely mixed state, because $\hat{\rho} \neq \frac{1}{2}\hat{\mathbb{1}}$ since $|\uparrow_{n_1}\rangle$ and $|\uparrow_{n_2}\rangle$ are not orthogonal.

Note: $|\uparrow_n\rangle\langle\uparrow_n| = \frac{1}{2}(\hat{\mathbb{1}} + \vec{e}_n\cdot\hat{\vec{\sigma}})$

$$\Rightarrow \hat{\rho} = \frac{1}{4}(\hat{\mathbb{1}} + \vec{e}_1\cdot\hat{\vec{\sigma}}) + \frac{1}{4}(\hat{\mathbb{1}} + \vec{e}_2\cdot\hat{\vec{\sigma}}) = \frac{1}{2}\left[\hat{\mathbb{1}} + \left(\frac{\vec{e}_1+\vec{e}_2}{2}\right)\right]\cdot\hat{\vec{\sigma}} = \frac{1}{2}\left[\hat{\mathbb{1}} + \frac{\vec{e}_z}{\sqrt{2}}\cdot\hat{\vec{\sigma}}\right]$$

$$\Rightarrow \hat{\rho} = \begin{bmatrix} \frac{1}{2}(1+\frac{1}{\sqrt{2}}) & 0 \\ 0 & \frac{1}{2}(1-\frac{1}{\sqrt{2}}) \end{bmatrix}$$ ← Same as in part (a)

Moral of the story: The ensemble decomposition is not unique. There are an infinite number of ensemble decompositions the correspond to the same $\hat{\rho}$.

(c) Consider two ensembles that are statistical mixtures of pure states

(I) $\{p_n, |\uparrow_n\rangle \mid n = 1, 2, 3, \cdots, N\}$

(II) $\{q_m, |\uparrow_m\rangle \mid m = 1, 2, 3, \ldots, M\}$

$$\hat{\rho}_I = \sum_{n=1}^{N} p_n |\uparrow_n\rangle\langle\uparrow_n| = \sum_{n=1}^{N} \frac{p_n}{2}(\hat{\mathbb{1}} + \vec{e}_n \cdot \hat{\vec{\sigma}}) = \frac{1}{2}\left(\hat{\mathbb{1}} + \left(\sum_{n=1}^{N} p_n \vec{e}_n\right) \cdot \hat{\vec{\sigma}}\right) = \frac{1}{2}(\hat{\mathbb{1}} + \vec{Q}_I \cdot \hat{\vec{\sigma}})$$

$$\hat{\rho}_{II} = \sum_{m=1}^{M} q_m |\uparrow_m\rangle\langle\uparrow_m| = \sum_{m=1}^{M} \frac{q_m}{2}(\hat{\mathbb{1}} + \vec{e}_n \cdot \hat{\vec{\sigma}}) = \frac{1}{2}\left(\hat{\mathbb{1}} + \sum_{m=1}^{M} q_m \vec{e}_m) \cdot \hat{\vec{\sigma}}\right) = \frac{1}{2}(\hat{\mathbb{1}} + \vec{Q}_{II} \cdot \hat{\vec{\sigma}})$$

$\hat{\rho}_I = \hat{\rho}_{II}$ iff their Bloch vectors are equal, $\vec{Q}_I = \vec{Q}_{II}$

$$\Rightarrow \sum_{n=1}^{N} p_n \vec{e}_n = \sum_{m=1}^{M} q_m \vec{e}_m$$

In part (a): $\hat{\rho}$ is statistical mixture of $|\uparrow_z\rangle$ $(\vec{e}_z)$ and $|\downarrow_z\rangle$ $(-\vec{e}_z)$

$$\vec{Q}_a = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)\vec{e}_z + \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)(-\vec{e}_z) = \frac{\vec{e}_z}{\sqrt{2}}$$

In part (b): $\hat{\rho}$ is a statistical mixture of $|\uparrow_{n_1}\rangle$ $(\vec{e}_{n_1})$ and $|\uparrow_{n_2}\rangle$ $(\vec{e}_{n_2})$

$$\vec{Q}_b = \frac{1}{2}\vec{e}_{n_1} + \frac{1}{2}\vec{e}_{n_2} = \frac{\vec{e}_z}{\sqrt{2}}$$

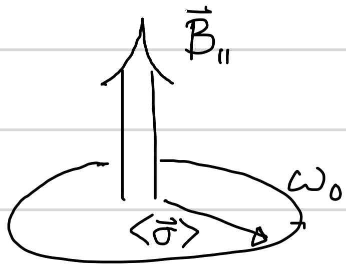
$\Rightarrow$ These are the same state! (Of course, I designed it that way by reverse engineering statistical mixture with the same Bloch vector).

Problem 3: Inhomogeneous Broadening

(a) Free induction decay by inhomogeneous broadening: Consider a macroscopic ensemble of spins in a static magnetic field in the z-direction with inhomogeneous magnitude described by a Gaussian distribution

$$P(B) = \frac{1}{\sqrt{2\pi\,\delta B^2}}\; e^{-(B-B_0)^2/2\delta B^2}$$

Suppose we apply a "hard" $\frac{\pi}{2}$-pulse that rotates the spin into the x-y plane. Each spin will precess about the z-axis at a frequency $\omega_0 = \gamma B$, depending on the local B-field. The rotating spin will radiate magnetic dipole radiation at frequency ω_0 in a long wave train.



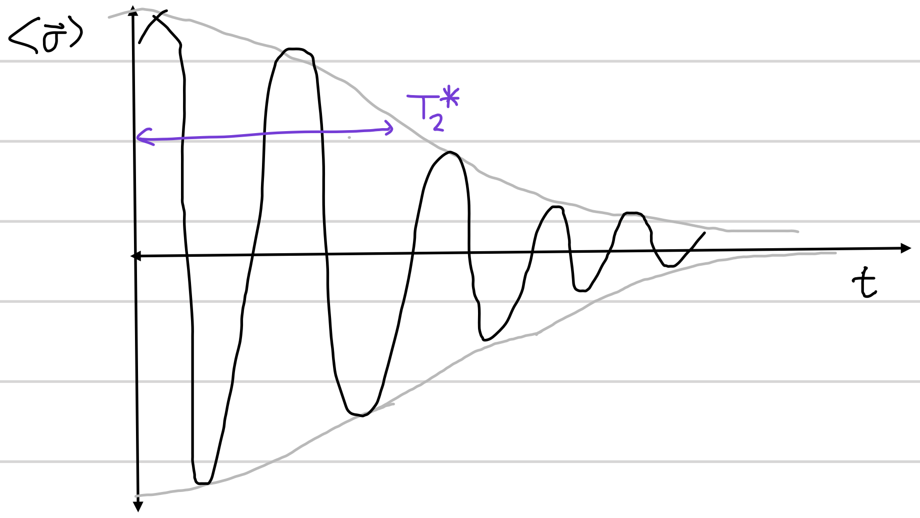
$$\langle\vec{\sigma}\rangle(t) = \langle\hat{\sigma}_x\rangle(0)\,\cos[\omega_0(B)t] + \langle\hat{\sigma}_y\rangle(0)\,\sin[\omega_0(B)t]$$
$$= \mathrm{Re}\left[\langle\hat{\sigma}_+(0)\rangle\, e^{-i\omega_0(B)t}\right]$$

Note: In order to achieve the "hard pulse", we must rotate all the spins by $\frac{\pi}{2}$. In the spin resonance $\Theta = \Omega_{tot}T = \frac{\pi}{2}$, where $\Omega_{tot} = \sqrt{\Omega^2 + \Delta^2}$. By choosing $\Omega >> |\Delta|$, for all Δ in the spread $\gamma\,\delta B$ then $\Omega_{tot} \approx \Omega$ and $\Theta \approx \frac{\pi}{2}$ for all spins in the ensemble. Thus, to achieve a hard pulse, choose a sufficient large Rabi frequency $\Omega << \frac{1}{T_2^*}$.

Because of the distribution of precession frequencies the different dipoles will radiate slightly different frequency waves. Eventually, these different waves will get out of phase with one another and so the total signal will show decay. This time scale is T_2^*.
The total signal is then the average of $\langle\vec{\sigma}\rangle(t)$ over the distribution of B-fields.
The spectrum of procession frequencies is equivalent to the distribution of B-fields.

$$\langle\vec{\sigma}\rangle_{tot}(t) = \mathrm{Re}\Big[\langle\hat{\sigma}_+(0)\rangle \underbrace{\int_{-\infty}^{\infty} dB\; P(B)\, e^{-i\omega_0(B)t}}_{\sim\ \text{Fourier transform}}\Big] \propto \mathrm{Re}\left(\langle\hat{\sigma}_+(0)\rangle\, e^{-\frac{(\delta B t)^2}{2}}\, e^{-i\gamma B_0 t}\right)$$

$$= e^{-\frac{(\delta B t)^2}{2}}\left(\langle\hat{\sigma}_x\rangle(0)\cos(\gamma B_0 t) + \langle\hat{\sigma}_y(0)\rangle\sin(\gamma B_0 t)\right)$$



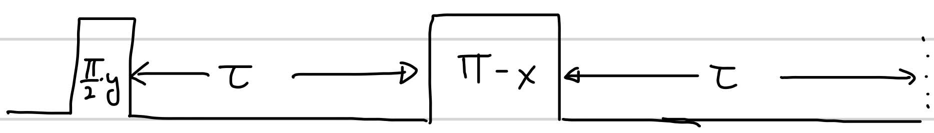
Wave train with a decaying Gaussian envelope due to inhomogeneity.

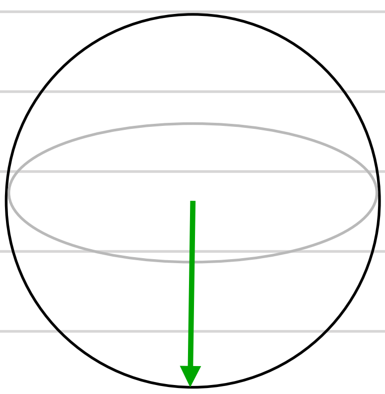
$$T_2^* = \frac{1}{\gamma(\delta B)} = \frac{1}{2\pi \times 10^{-2}} \text{ s}$$

$$T_2^* \approx 16\ \mu s.$$

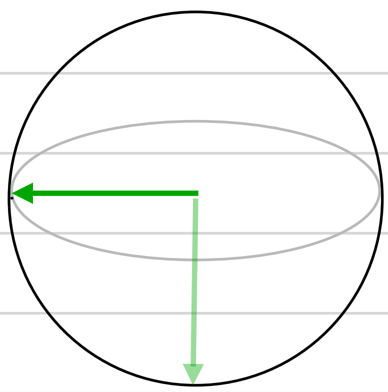
(b) Spin echo: Time reversing inhomogeneous (but coherent) evolution

Consider the following sequence: $\frac{\pi}{2}$-pulse about y – free evolution τ – π-pulse around x – free evolution τ

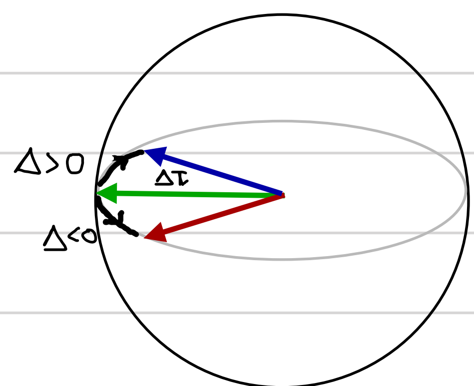




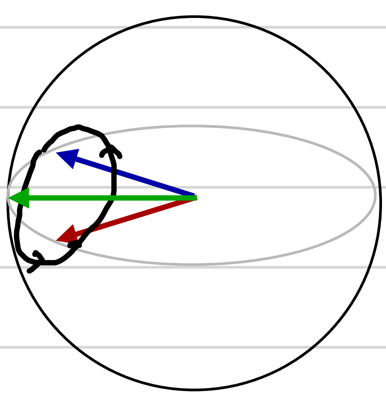
Initially all spins down



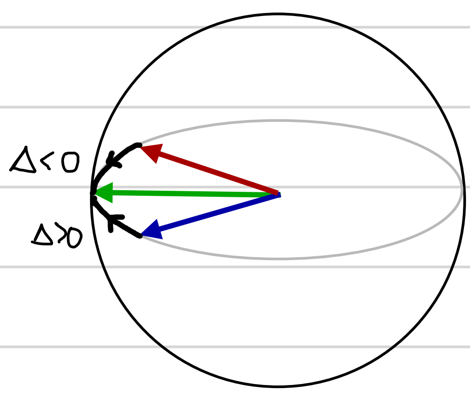
A hard $\frac{\pi}{2}$ x pulse rotates to y-axis



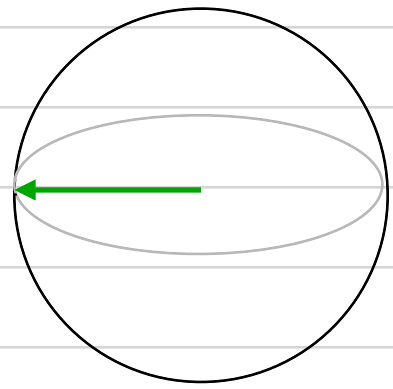
Spins freely precess spread out due to inhomogeneous detunings



A π pulse around y flips the red and blue detuned

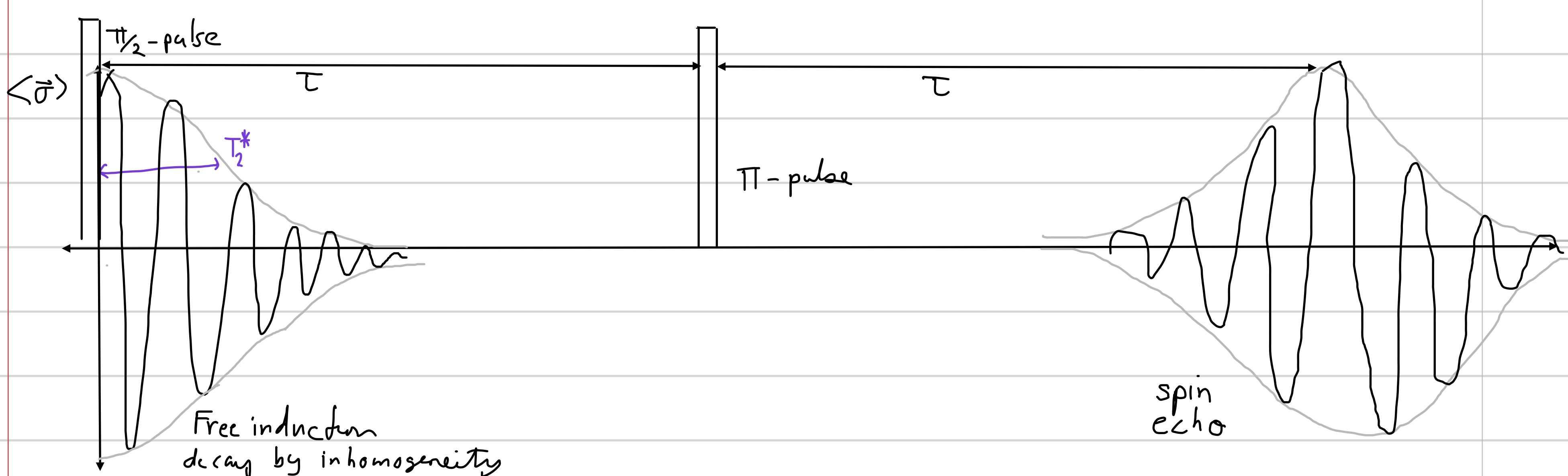


Everything is now time reversed and the spreading "refocuses"



After time τ we rephase to echo!

The key idea of the spin echo is that the π pulse effectively flips $z \Rightarrow -z$. The same spread in B_z leads to the same spread in precession frequencies, but in the reverse direction. We effectively time-reverse the spread of precession angles of the Bloch vectors. If a given Bloch vector precessed an angle $\Delta\tau$ then after the π pulse it precesses $-\Delta\tau$. After this time, all the Bloch vectors return to the y axis. Note, in this problem, we have chosen the $\frac{\pi}{2}$ pulse and the echo pulse around orthogonal axes. The same echo phenomenon would occur if we had chosen the sequence $\frac{\pi}{2}x - \tau - \pi_x - \tau$. In that case the Bloch vector would refocus on the $-y$-axis, but otherwise, everything is the same.



The Ramsey interferometer is intimately related to a "two-path" wave interferometer, such as a Mach-Zender interferometer. The spin-echo pulse is equivalent to flipping the two modes with mirrors. The detuning between the applied oscillating "transverse field" and the "longitudinal" resonance frequency plays the role of the frequency in the Mach-Zender interferometer. The MZ-interferometer is robust to the frequency of the light. That is, if the two arms are perfectly balanced, we achieve perfect interference, regardless of the frequency of light. Such an interferometer is known as a "white-light" interferometer, because it will shown coherent interference for a broad spectrum of frequencies.